

Stability criteria for q -expectation values

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Abstract

In statistical physics lately a specific kind of average, called the q -expectation value, has been extensively used in the context of q -generalized statistics dealing with distributions following power-laws. In this context q -expectation values appear naturally. After it has been recently shown that this non-linear functional is instable, under a very strong notion of stability, it is therefore of high interest to know sufficient conditions for when the results of q -expectations are robust under small variations of the underlying distribution function. We show that reasonable restrictions on the domain of admissible probability distributions restore uniform continuity for the q -expectation. Bounds on the size of admissible variations can be given. The practical usefulness of the theorems for estimating the robustness of the q -expectation value with respect to small variations is discussed.

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I. INTRODUCTION

In the context of generalizations of entropy-functionals generalized momenta occur naturally [1], which, in the case of Tsallis q -statistics [2] are commonly called escort distributions. Aside from their necessity in several aspects of q -statistics, expectation values under these escort distributions have been used to replace ordinary constraints in the maximum entropy principle [3]. Maximizing under these escort constraints (also called q -constraints) via functional variations with respect to distributions p , the classical Tsallis entropy, $S_q = - \int p \ln_{2-q} p$, produces the famous q -exponential distributions, where the q -exponential function is defined as $\exp_q(x) = (1 + (1 - q)x)^{1/(1-q)}$. However, note that in general there is no need for q -constraints in the Tsallis formalism; the same q -exponential distributions can be derived under ordinary constraints when Tsallis entropy is expressed in its dual form, $S_q = - \int p \ln_q p$, see [4]. The way generalized momenta still occur is when differential properties of ordinary expectation values are considered [1]. For example, one may look at the q -exponential distribution $\exp_q(-\alpha - \beta\epsilon_i)$, where ϵ_i are discrete energy states, β is the inverse temperature and α is used for normalization, i.e. the normalization condition $1 = \sum_i \exp_q(-\alpha - \beta\epsilon_i)$ holds. The way α has to change with β , in this case, can be obtained by differentiating the normalization condition with respect to β and using $d \exp_q(x)/dx = \exp_q(x)^q$. Therefore,

$$\frac{d\alpha}{d\beta} = - \frac{\sum_i \exp_q(-\alpha - \beta\epsilon_i)^q \epsilon_i}{\sum_i \exp_q(-\alpha - \beta\epsilon_i)^q} \quad , \quad (1)$$

where the right side exactly corresponds to the q -expectation value

$$\langle \epsilon \rangle_q \equiv \frac{\sum_i p_i^q \epsilon_i}{\sum_i p_i^q} \quad , \quad (2)$$

when $p_i = \exp_q(-\alpha - \beta\epsilon_i)$ is the q -exponential distribution. The distribution

$$P_i^{(q)} = \frac{p_i^q}{\sum_i p_i^q} \quad (3)$$

usually is called the escort distribution of p . One should note that, with respect to p , q -expectation values

$$\langle \mathcal{O} \rangle_q \equiv \sum_i P_i^{(q)} \mathcal{O}_i \quad , \quad (4)$$

of some observables $\mathcal{O} = \{\mathcal{O}_i\}$ are non-linear functionals. In the entire paper we refer to the q -expectation value as a functional and will use the notation $Q[p] \equiv \langle \mathcal{O} \rangle_q$, to show its

explicit dependency on p . For all mathematical notions that will be used in this paper, like for instance equicontinuity, uniform continuity or Lebesgue decomposition, we refer to standard textbooks on functional analysis, like e.g. [5].

It has to be noted that the question of continuity of functionals has been of some interest lately, see e.g. [6, 7, 8, 9, 10]. Recently, it has been shown that, under small variations of the probability distribution, q -expectation values are instable in a certain sense [11]. It was concluded there that, due to this certain lack of stability, the usage of q -expectation values should be reconsidered in q -statistical physics. Therefore, it is important to ask whether this argument really disqualifies the usage of q -expectation values in general.

The notion of stability used in [11] is closely related to stability in the sense of Lesche [6]. Let us write probabilities p on \mathbb{N} such that $\sum_i p_i = 1$ and the $\|p\|_1 = \sum_i |p_i|$ is the $L_1(\mathbb{N})$ -norm. Probabilities on finite sets $i = 1 \dots W$ will simply be represented on \mathbb{N} with $p_i = 0$ for all $i > W$, as in [10].

In [11] a functional $F[p]$ is called stable, if for all $\epsilon > 0$ there exists a $\delta > 0$, such that for all sequences of probabilities $p \{p_W\}_{W=1}^\infty$ and $\{p'_W\}_{W=1}^\infty$, where $p_{W,i} = 0$ and $p'_{W,i} = 0$ for all $i > W$, it is true that

$$\forall W [\|p_W - p'_W\|_1 < \delta] \Rightarrow |F[p_W] - F[p'_W]| < \epsilon \quad . \quad (5)$$

Defining

$$\mathcal{D}_0 = \{ \{p_W\}_{W=1}^\infty \mid \forall W [\|p_W\|_1 = 1], \forall i > W [p_{W,i} = 0] \} \quad , \quad (6)$$

the same definition, Eq. (5) can be formulated shortly by calling a functional $F[p]$ stable if it is uniformly equicontinuous on \mathcal{D}_0 . To prove instability of a functional F on \mathcal{D}_0 it is sufficient to find one example of a sequences $\{p_W\}_{W=1}^\infty \in \mathcal{D}_0$, such that uniform equicontinuity of the functional F is violated. This is exactly what has been done in [11]. Two examples, one for $0 < q < 1$ and one for $1 < q$, which originally have been used by Lesche [6] (for a detailed discussion see e.g. [10]), show that the q -expectation value $Q[p]$ is not uniformly equi-continuous on \mathcal{D}_0 and therefore prove that $Q[p]$ is not stable. The recognition of such instabilities is important, since they point at the fact that, under certain conditions or under certain circumstances, it will be difficult to correctly estimate reliable values of $Q[p]$ (or any other functional, for instance entropies, that possesses an instability; see e.g. [10]).

On the other hand, properties, like uniform continuity, are not simply properties of a functional but are properties of a functional together with a domain of definition. Identification of the problematic regions, in the domain of definition of the functional, therefore provides information on where on its domain the functional can be used without running into the instabilities the functional potentially possesses. In the context of functions \sqrt{x} may serve as an example. \sqrt{x} is uniformly continuous on all intervals $[a, b]$, with $0 < a < b < \infty$, but is not uniformly continuous on some interval $[0, b]$. Uniform continuity fails when 0 is an element of the considered domain of \sqrt{x} . Similarly, we may ask whether reasonable domains $\mathcal{D} \subset \mathcal{D}_0$ can be found, such that the functional F is uniformly equi-continuous on \mathcal{D} , even though the functional is not uniformly equicontinuous on \mathcal{D}_0 . If such a \mathcal{D} exists we can call F stable on \mathcal{D} . We will show in this paper that it is possible to find domains \mathcal{D} , such that the q -expectation value $Q[p]$ is stable with respect to \mathcal{D} as a functional. Moreover, the domains \mathcal{D} are large enough to contain a large range of situations that usually are of physical interest. This will show that for this range of practical situations the q -expectation value can safely be used and small variations of the distribution functions will not lead to uncontrollable variations of the associated q -expectation values. The stability question in the case of q -expectation values is especially of interest as, for instance, it has been shown that a variety of correlated processes may lead to limit distributions that are extremely close to q -exponential functions but are not q -exponential functions after all [12]. If in an effective theory experimental data should for practical means be misinterpreted in terms of q -exponential functions it therefore is crucial to know how reliable the predictions will be, given the experimental uncertainty with respect to the underlying distribution.

In order to understand the instability let us take a look at the two examples [6, 11] violating uniform equi-continuity of the q -expectation value $Q[p]$, where case (1) is associated with $0 < q < 1$, and case (2) with $1 < q$. Specifically, in [11] the two cases are

case (1): $0 < q < 1$

$$p_i = \delta_{i1} \quad , \quad p'_i = \left(1 - \frac{\delta}{2} \frac{W}{W-1}\right) p_i + \frac{\delta}{2} \frac{1}{W-1} \quad (7)$$

case (2): $1 < q$

$$p_i = \frac{1}{W-1} (1 - \delta_{i1}) \quad , \quad p'_i = \left(1 - \frac{\delta}{2}\right) p_i + \frac{\delta}{2} \delta_{i1} \quad , \quad (8)$$

where obviously $\|p - p'\|_1 = \delta$, for any finite W . In the limit $W \rightarrow \infty$ both cases lead to $\lim_{W \rightarrow \infty} |Q[p] - Q[p']| = |\bar{\mathcal{O}} - \mathcal{O}_1|$, where $\bar{\mathcal{O}} \equiv \lim_{W \rightarrow \infty} W^{-1} \sum_i \mathcal{O}_i$, which proves instability

on \mathcal{D}_0 when \mathcal{O} and K are chosen such that $|\bar{\mathcal{O}} - \mathcal{O}_1| > K > 0$. This is true, since in the limit there already exists a W_0 such that $|Q[p] - Q[p']| > K$ for all $W > W_0$.

Though, this is not necessary for the validity of this proof one may note that the considered sequences of probabilities have a limit that is not a probability, i.e. the limit $W \rightarrow \infty$ and the $L_1(\mathbb{N})$ -norm do not commute. For instance, $\delta = \lim_{W \rightarrow \infty} \sum_{i=1}^W |p_i - p'_i| \neq \sum_{i=1}^{\infty} \lim_{W \rightarrow \infty} |p_i - p'_i| = \delta/2$, in both cases. This means that p or p' are in general not probabilities in the pointwise limit, e.g., in case $0 < q < 1$, one gets $\sum_i \lim_{W \rightarrow \infty} p'_i = (1 - \delta/2) \neq 1$. The considered sequences of probabilities $\{p_W\}_{W=1}^{\infty}$ can easily be interpreted as a limit to distributions $\rho(x)$ on the continuous interval $x \in I \equiv [0, 1]$ with $\int_I dx \rho = 1$, where dx is the usual Lebesgue measure on $[0, 1]$. We will analyze the stability problem within this continuous formulation. This is done for two reasons. First, the continuity properties of q -expectation values with respect to distribution functions $\rho(x)$, where $x \in [0, 1]$, are of interest on their own, since power distributions are not limited to discrete state spaces. The second reason is that the discrete case is naturally embedded in the continuous case, as demonstrated below. Propositions obtained in the continuous case can therefore be used to discuss continuity properties of the q -expectation value in the discrete case. In the continuous case we will denote the q -expectation with $\tilde{Q}[\rho] \equiv \int dx \rho^q(x) \mathcal{O}(x) / \int dx \rho^q(x)$, where the observable $\mathcal{O}(x)$ now is suitable measurable function on $[0, 1]$.

A. The problem formulated for continuous distributions

The problem of the ill defined limit probabilities of the examples (1) and (2) is easily resolved by mapping the discrete probabilities $\{p_{W,i}\}_{i=1}^W$ onto step functions $\rho_W(x)$, with $x \in [0, 1]$ such that $\rho_W(x) = W p_i$ for $x \in [(i-1)/W, i/W)$ (the last interval is chosen $[(W-1)/W, 1]$). Therefore, for the usual Lebesgue measure dx on $[0, 1]$ it follows that $\int_0^1 dx \rho_W(x) = \sum_{i=1}^W \int_{(i-1)/W}^{i/W} \rho_W(x) = \sum_{i=1}^W p_{W,i} = 1$, and the $L_1(\mathbb{N})$ -norm $\|\rho - \rho'\|_1 = \int dx |\rho(x) - \rho'(x)| = \sum_i |p_i - p'_i| = \|p - p'\|_1$. Similarly, the discrete observable \mathcal{O} is mapped to a step function in an analogous way by identifying $\mathcal{O}_i = \mathcal{O}(x)$ when $x \in [(i-1)/W, i/W)$. The discrete and the continuous q -expectation value therefore coincide since $\tilde{Q}[\rho] = \int dx \rho^q(x) \mathcal{O}(x) / \int dx \rho^q(x) = \sum_i p_i^q \mathcal{O}(x) / \sum_i p_i^q \equiv Q[p]$. In this way the limit $W \rightarrow \infty$ can be interpreted as the continuum limit of the step-functions ρ and ρ' . These limits are well-defined probability distributions, and the L_1 -norm of the distributions and the $W \rightarrow \infty$

limit commute.

In this continuum formulation the limit distributions of the families of distributions examples, case (1) and (2), that have violated uniform equi-continuity are given by

case (1): $0 < q < 1$

$$\rho(x) = \delta(x) \quad , \quad \rho'(x) = (1 - \frac{\delta}{2})\delta(x) + \frac{\delta}{2} \quad (9)$$

case (2): $1 < q$

$$\rho(x) = 1 \quad , \quad \rho(x)' = 1 - \frac{\delta}{2} + \frac{\delta}{2}\delta(x) \quad , \quad (10)$$

where $\delta(x)$ is the usual delta function. The result of [11], that in the limit $W \rightarrow \infty$, $|Q[p] - Q[p']| = |\bar{\mathcal{O}} - \mathcal{O}_1| > 0$ for $\|p - p'\| = \delta$, in the continuum translates into that

$$|\tilde{Q}[\rho] - \tilde{Q}[\rho']| = |\mathcal{O}_1 - \bar{\mathcal{O}}| \quad , \quad (11)$$

for $\|\rho - \rho'\| = \delta$, with $\mathcal{O}_1 = \mathcal{O}(0)$ and $\bar{\mathcal{O}} \equiv \int dx \mathcal{O}(x)$. Therefore, the first requirement we have to impose on \mathcal{D} is, that the sequences $\{p_W\}_{W=1}^\infty \in \mathcal{D}$ possess a continuum limit in $[0, 1]$ with respect to the 1-norm on $L_1([0, 1])$. Let us denote the set of all limit distributions produced by the sequences in \mathcal{D} with $\tilde{\mathcal{D}}$. If uniform equicontinuity of $Q[p]$ with respect to \mathcal{D} has to hold it is therefore necessary that $\tilde{Q}[\rho]$ is uniformly continuous on $\tilde{\mathcal{D}}$. This serves as starting point of the analysis.

The rest of the paper is organized as follows. In section II we present two theorems for the cases (1) $0 < q < 1$ and (2) $1 < q$ that allow to analyze the continuity of $\tilde{Q}[\rho]$ around the distribution ρ . The bounds given in the theorems are such that obvious definitions of the domain $\tilde{\mathcal{D}}$ of $\tilde{Q}[\rho]$ guarantees uniform continuity of the q -expectation value $\tilde{Q}[\rho]$ on these domains. An upper bound of admissible variations on these domains is discussed which can be seen as a measure of overall robustness of the q -expectation values on these domains, which may provide a practical mean to check experimental situations for their robustness. In the discussion III we will show how the theorems can be used in two examples. First, we will discuss there how the properties of $\tilde{\mathcal{D}}$ can be pulled back to a suitable \mathcal{D} so that the q -expectation value $Q[p]$ becomes uniform equicontinuous on \mathcal{D} . Second, we will briefly discuss how the theorems can be used to analyze the continuity properties of $\tilde{Q}[\rho]$ for distributions defined on the infinite interval $[0, \infty]$. This result allows to consider a different subclass of $\mathcal{D}' \subset \mathcal{D}_0$ where sequences of probabilities p in $\mathcal{D}' \subset \mathcal{D}_0$, possess a limit with respect to the 1-norm on $L_1(\mathbb{N})$ and where the 1-norm on $L_1(\mathbb{N})$ and the limit $W \rightarrow \infty$ commute.

II. THE INSTABILITY IN THE GENERAL CASE

In the continuum the escort distribution reads $P^{(q)}(x) \equiv \frac{\rho(x)^q}{\int dx' \rho(x')^q}$. The expectation value of a function $\mathcal{O}(x)$ under this measure – the q -expectation value – is then $\tilde{Q}[\rho] = \int dx P^{(q)}(x) \mathcal{O}(x)$. The total variation of $\tilde{Q}[\rho]$ therefore reads

$$\delta \tilde{Q}[\rho] = \tilde{Q}[\rho + \delta \rho] - \tilde{Q}[\rho] \quad . \quad (12)$$

We can now analyze the two cases separately. The following proofs are carried out on the unit interval $I \in [0, 1]$. This does not present a loss of generality, since the proofs can be extended to any bounded interval. For unbounded intervals, especially relevant for $q > 1$, the proofs get more involved and require to fix conditions that relate to specific boundedness conditions for the observable and decay properties of ρ , in order to keep $\tilde{Q}[\rho]$ a meaningful quantity as is briefly discussed in section III.

A. The case $0 < q < 1$

Looking at Equ. (9) one can suppose that the uniform continuity property of the q -expectation value $\tilde{Q}[\rho]$ is discontinuous for $\rho(x) = \delta(x)$ since is a pure point measure. Due to Lebesgue decomposition for distributions each distribution ρ can be decomposed into a singular part ρ_s , that is defined on a set of Lebesgue measure zero, and an absolute continuous part ρ_c . We therefore assume that the distribution ρ in the theorem is not purely singular, i.e. it possesses an absolute continuous part ρ_c with $\int_I dx \rho_c > 0$. Note that $\|f\|_p = (\int_I dx |f|^p)^{1/p}$ is the usual p -norm on $I = [0, 1]$ and $\|f\|_\infty = \sup\{|f(x)| \mid x \in [0, 1]\}$.

In order to prove Theorem (1), we have to establish propositions 1 - 8, see Appendix A.

Theorem 1. *Let $0 < q < 1$. Let $\tilde{Q}[\rho] \equiv \langle \mathcal{O} \rangle_q$ be the associated q -expectation value for the observable \mathcal{O} . Let the distribution $0 < \rho$ on $I = [0, 1]$ have a non vanishing absolute continuous (non-singular) part. Let $G = \int_I dx \rho(x)^q$ and let $0 < \tilde{\delta}^q = \mu G/4$ for $0 < \mu < 1$ and $\delta \rho$ be a variation of the distribution such that $\int_I dx |\delta \rho| = \delta \leq \tilde{\delta}$, and $0 < \rho + \delta \rho$ is positive on I . Furthermore let $0 < \mathcal{O}$ be a strictly positive bounded observable on I , then there exists a constant $0 < c < \infty$, such that*

$$|\tilde{Q}[\rho] - \tilde{Q}[\rho + \delta \rho]| < c \delta^q \quad . \quad (13)$$

Moreover $c \leq 4G^{-2} \|\mathcal{O}\|_\infty (1 + \|\mathcal{O}\|_\infty \|\mathcal{O}^{-1}\|_\infty) / (1 - \mu)$.

Proof. The requirement that ρ is not purely singular is sufficient to guarantee that $0 < \int_I dx \rho^q \mathcal{O}$ is strictly positive. Inversely, suppose $\rho(x) = \delta(x - x_0)$ is concentrated around one point x_0 and use the characteristic function $\Delta^{-1} \chi_{[-\frac{\Delta}{2}, \frac{\Delta}{2}]}(x - x_0)$ as a δ -sequence. The characteristic function $\chi_{[a,b]}(x) = 1$ for $x \in [a, b]$ and zero otherwise. It is straight forward to see that $\int_{D_+} dx \rho(x)^q = \int_0^\Delta dx \Delta^{-q} = \Delta^{1-q} \rightarrow 0$, for $\Delta \rightarrow 0$. This can not happen if ρ has a non vanishing absolutely continuous part.

Note that

$$|\tilde{Q}[\rho] - \tilde{Q}[\rho + \delta\rho]| = |\tilde{Q}[\rho]| |1 - \tilde{Q}[\rho + \delta\rho] \tilde{Q}[\rho]^{-1}| \quad . \quad (14)$$

Using the Hölder-inequality one finds $\int_I dx \rho(x)^q \mathcal{O}(x) \leq \|\mathcal{O}\|_{1/(1-q)} \leq \|\mathcal{O}\|_\infty$. Consequently $|\tilde{Q}[\rho]| < \|\mathcal{O}\|_\infty / G$. Furthermore, note that $\int_I dx \rho(x)^q \mathcal{O}(x) \geq G / \|\mathcal{O}^{-1}\|_\infty$. Propositions (1-8) imply that

$$\left(1 - \frac{\tilde{C}_2 \delta^q}{G - \tilde{C}_3 \tilde{\delta}^q}\right) \left(1 - \frac{C_3 \delta^q}{\int_I dx \rho^q \mathcal{O}}\right) \leq \frac{\tilde{Q}[\rho + \delta\rho]}{\tilde{Q}[\rho]} \leq \left(1 + \frac{C_2 \delta^q}{\int_I dx \rho^q \mathcal{O}}\right) \left(1 + \frac{\tilde{C}_3 \delta^q}{G - \tilde{C}_3 \tilde{\delta}^q}\right) \quad (15)$$

Setting the constants to their upper bounds, i.e. $C_2 \rightarrow 4\|\mathcal{O}\|_\infty$, $C_3 \rightarrow 4\|\mathcal{O}\|_\infty$, $\tilde{C}_2 \rightarrow 4$, $\tilde{C}_3 \rightarrow 4$, and evaluating the terms of the left and the right side gives $(1 - a_1 \delta^q + a_2 \delta^{2q}) \leq \tilde{Q}[\rho + \delta\rho] / \tilde{Q}[\rho] \leq (1 + b_1 \delta^q + b_2 \delta^{2q})$ and the resulting constants a_1 , a_2 , b_1 , and b_2 are all positive. On the left side we note that $1 - a_1 \delta^q \leq 1 < 1 - a_1 \delta^q + a_2 \delta^{2q}$ and on the right side $1 + b_1 \delta^q + b_2 \delta^{2q} < 1 + b_1 \delta^q + b_2 \tilde{\delta}^q \delta^q$. Furthermore, $a_1 < b_1 + b_2 \tilde{\delta}^q$. This allows to give an upper bound for c given by $c = \|\mathcal{O}\|_\infty (b_1 + b_2 \tilde{\delta}^q) / G$. Moreover, $b_1 = (4/(1-\mu) + 4\|\mathcal{O}\|_\infty \|\mathcal{O}^{-1}\|_\infty) / G$ and $b_2 \tilde{\delta}^q = 4G^{-1} \|\mathcal{O}\|_\infty \|\mathcal{O}^{-1}\|_\infty \mu / (1 - \mu)$ which completes the proof. \square

The theorem (together with its associated propositions) states that for strictly positive bounded observables q -expectation values are continuous for non purely singular ρ , i.e. the absolute continuous part of ρ is non vanishing. Clearly uniform continuity of the q -expectation value can not be established on all of $L_1([0, 1])$. However, it follows from Theorem (1) that on any domain

$$\tilde{D}_{B,r}^{(1)} = \{\rho | 0 < \rho \in L_1([0, 1]), \|\rho\|_1 = 1, 0 \leq r \leq \int_I dx \rho(x)^q \leq B\} \quad (16)$$

the q -expectation value $\tilde{Q}[\rho]$ is uniformly continuous. The lower bound r on $\int_I dx \rho(x)^q$ is required in order to exclude distributions with purely singular measure[13]. The constant c in general is depending on ρ since $G = \int_I dx \rho(x)^q$. However due to the common lower

bound r it follows that $G \geq r$ on all $\rho \in \tilde{\mathcal{D}}_{B,r}$. Therefore, choosing $c = 4r^{-2}||\mathcal{O}||_\infty(1 + ||\mathcal{O}||_\infty||\mathcal{O}^{-1}||_\infty)/(1 - \mu)$ is a sufficiently large on all of $\rho \in \tilde{\mathcal{D}}_{B,r}$ and c does not depend on the particular choice of $\rho \in \tilde{\mathcal{D}}_{B,r}$ any more. Consequently, uniform continuity of the q -expectation value $\tilde{Q}[\rho]$ is established on any domain $\rho \in \tilde{\mathcal{D}}_{B,r}$. Further, since $\tilde{\delta} = (\mu G/4)^{1/q}$ is an upper bound on the L_1 -norm $\delta = ||\delta\rho||_1$ of variations $\delta\rho = \rho - \rho'$, guaranteeing the validity of $|\tilde{Q}[\rho] - \tilde{Q}[\rho']| \leq c\delta$. Most likely these bound can be improved. Yet, $\tilde{\delta}$ can be seen as a measure of robustness of the q -expectation value $\tilde{Q}[\rho]$ on $\tilde{\mathcal{D}}_{B,r}$. To make $\tilde{\delta}$ independent of the choice of ρ one has to set $\tilde{\delta} = (\mu B/4)^{1/q}$. It has to be noted that the upper bound $\tilde{\delta}$ decreases with increasing B like $0 < \tilde{\delta}^q = \mu B^{q-1}/4$ and therefore robustness under variations will in general decrease with increasing B .

We want to remark that the condition of strict positivity of the observable, we have required as a condition in the theorem, can be relaxed to observables that are bounded from below by some constant L , i.e. $\mathcal{O} \geq L > -\infty$. If this is the case, one can look at the observable $\mathcal{O}_L = \mathcal{O} - L + 1$, which is strictly positive and $||\mathcal{O}_L^{-1}||_\infty = 1$. Since for the q -expectation value it is true that $\langle 1 \rangle_q = 1$ for any admissible distribution it is also true that $\langle \mathcal{O}_L \rangle_q = \langle \mathcal{O} \rangle_q - L + 1$. The results therefore relax to bounded observables, i.e. $||\mathcal{O}||_\infty < \infty$. By shifting \mathcal{O} to \mathcal{O}_L we can make the substitutions in the bounds $||\mathcal{O}_L||_\infty \rightarrow 2||\mathcal{O}||_\infty + 1$ and $||\mathcal{O}_L^{-1}||_\infty \rightarrow 1$

B. The case $1 < q$

In contrast to the $0 < q < 1$ case, the instability in the $1 < q$ case is not caused by purely singular distributions ρ , but due to the variation $\delta\rho$ having a non vanishing singular part. In order to prove Theorem (2), we have to establish propositions 9 - 14, see Appendix B.

Theorem 2. *Let $q > 1$ and let $m > 0$ be an arbitrary but fixed constant. Let $0 < \rho$ be a probability distribution on $I = [0, 1]$. Let $\delta\rho$ be variations of ρ , i.e. $\rho + \delta\rho > 0$. Let $\tilde{Q}[\rho] = \langle \mathcal{O} \rangle_q$ be the q -expectation value and let $0 < \mathcal{O}$ be a strictly positive bounded observable on I . Let $B > 0$ be an arbitrary but fixed constant. Let the variations $\delta\rho$ be uniformly bounded in the m -norm, such that $||\delta\rho||_m < B$. Further let $||\delta\rho||_1 = \delta$. Let $\tilde{\delta}$ be an upper bound for the size of the variations δ such that $(2^{1/q} - 1)^{q/\gamma} B^{(\gamma-q)/\gamma} (\min(1, ||\mathcal{O}||_\infty ||\mathcal{O}^{-1}||_\infty))^{-1/\gamma} \geq \tilde{\delta} > 0$,*

where $\gamma = (m - q)/(m - 1)$. Then, there exists a constant $0 < R < \infty$, such that

$$|\tilde{Q}[\rho] - \tilde{Q}[\rho + \delta\rho]| < R\delta^{\gamma/q} \quad , \quad (17)$$

and R does not depend on the choice of ρ .

Proof. This result follows directly from propositions (9-14) from Appendix B, and by noting that

$$\frac{1 - R_2\delta^{\gamma/q}}{1 + \tilde{R}_2\delta^{\gamma/q}} \leq \frac{\tilde{Q}[\rho + \delta\rho]}{\tilde{Q}[\rho]} \leq \frac{1 + R_2\delta^{\gamma/q}}{1 - \tilde{R}_2\delta^{\gamma/q}} \quad . \quad (18)$$

Proposition 14 tells us that $1/(1 - \tilde{R}_2\delta^{\gamma/q}) \leq 1 + R_3\delta^{\gamma/q}$. Moreover $1/(1 + \tilde{R}_2\delta^{\gamma/q}) \geq 1 - \tilde{R}_2\delta^{\gamma/q}$. Note that $R_3 > \tilde{R}_2$. Since $\|\mathcal{O}^{-1}\|_{\infty}^{-1} \leq \tilde{Q}[\rho] \leq \|\mathcal{O}\|_{\infty}$ choosing $R = R_2R_3 \max\{\|\mathcal{O}^{-1}\|_{\infty}^{-1}, \|\mathcal{O}\|_{\infty}\}$ is sufficient. Noting that both R_2 and R_3 are not depending on the particular choice of ρ completes the proof. \square

The theorem (together with its associated propositions) states that, for strictly positive bounded observables, q -expectation values are continuous for any ρ , as long as the variation $\delta\rho = \rho' - \rho$ is bounded in some m -norm with $m > q$. By considering domains

$$\tilde{\mathcal{D}}_{B,m}^{(2)} = \{\rho | \rho \in L_1([0, 1]) \cap L_m([0, 1]), \|\rho\|_m \leq B\} \quad (19)$$

for case (2), i.e. $1 < q$, automatically any admissible variation $\|\delta\rho\|_m < B$ and the constant R is not depending on the particular choice of admissible variation with respect to the domain $\tilde{\mathcal{D}}_{B,m}$ any more. This proves that the q -expectation value $\tilde{Q}[\rho]$ is uniformly continuous on any $\tilde{\mathcal{D}}_{B,m}$ with $m > q$. Again, it has to be noted that $\tilde{\delta} \propto B^{(\gamma-q)/\gamma}$. Since $0 < \gamma < 1$ and $q > 1$ it follows that $(\gamma - q)/\gamma < 0$ and $\tilde{\delta}$ decreases as B increases. Measuring robustness in $\tilde{\delta}$ again shows that robustness of the q -expectation value with respect to small variations decreases with enlarging the domain of definition as expected.

III. DISCUSSION

We will now demonstrate the practicability of the two theorems by discussing two applications of the theorems. The first application is to understand when uniform equicontinuity of families of sequences of probabilities can be expected. The second application is to extend the conditions for uniform continuity of the q -expectation value from the case where

the distributions have compact support, i.e. $[0, 1]$, to the case where distributions have an unbounded support $[0, \infty]$.

First, we turn to the question of uniform equicontinuity the q -expectation value. We have shown in section II that q -expectation values are uniformly continuous for domains that in case (1) $0 < q < 1$ have been specified in Eq. (16) and in case (2) $1 < q$ in Eq. (19). These results allow to establish equicontinuity properties of q -expectation values for sequences of probabilities $\{p_W\}_{W=1}^\infty \in \mathcal{D} \subset \mathcal{D}_0$, specified in Eq. (6).

To make the contact with the continuum results, we have to impose that the limit of the sequences of probabilities in \mathcal{D} exists as continuum limits in the $L_1([0, 1])$ -norm, i.e. in terms of step functions ρ_W representing p_W , as described in section I A. The span of these limits has to coincide with the domain $\tilde{\mathcal{D}}$. This can be achieved when all the distributions $\rho_W \in \tilde{\mathcal{D}}$. In case (1) the conditions defining $\tilde{\mathcal{D}}_{B,r}$, for some $0 < r < B$, translate into the requirement that

$$r \leq W^{q-1} \sum_i p_{W,i}^q \leq B \quad . \quad (20)$$

In case (2) the conditions defining $\tilde{\mathcal{D}}_{B,m}$, for some $m > q > 1$ and some $B > 0$, translate into the requirement that

$$\|p_W\|_m \leq BW^{\frac{1-m}{m}} \quad . \quad (21)$$

By imposing these conditions on the domain of sequences \mathcal{D} , uniform equicontinuity of the q -expectation value, with respect to \mathcal{D} , can be established for both cases (1) and (2). Consequently, q -expectation values can be called *robust* or *stable* with respect to the specified domains \mathcal{D} .

We discuss a second application of the theorems, to establish criteria for specifying subsets of probability distributions $\rho \in L_1([0, \infty])$ such that the q -expectation value again is uniformly continuous on this domain. Again, one can use the results of section II as a starting point of the discussion and proceed as follows.

Choose a suitable differentiable monotonous functions, $g : [0, \infty] \mapsto [0, 1]$. Let g' denote the derivative of g and g^{-1} the inverse function of g . Therefore, g maps the distribution function ρ , defined on $[0, \infty]$, to a distribution function $\tilde{\rho}(y) = \rho(g^{-1}(y))g'(g^{-1}(y))^{-1}$ on $[0, 1]$. Similarly, the observable function \mathcal{O} on $[0, \infty]$ gets mapped to $\tilde{\mathcal{O}}(y) = \mathcal{O}(g^{-1}(y))g'(g^{-1}(y))^{q-1}$. Applying the conditions used for the theorems 1 and 2 and characterizing domains where the q -expectation value on $[0, 1]$ is uniformly continuous poses

restrictions on the transformed distributions $\tilde{\rho}$ and the transformed observable $\tilde{\mathcal{O}}$. These restrictions can now be pulled back to the distribution ρ and the observable \mathcal{O} on $[0, \infty]$. For specific problems different choices of g may be considered. It is instructive to look at an explicit example. Consider \bar{q} -exponential distributions $\rho(x) \propto e_{\bar{q}}(-\beta x) \equiv [1 - (1 - \bar{q})\beta x]^{\frac{1}{1-\bar{q}}}$ for $\bar{q} \geq 1$ and some inverse temperature β . Assume that we wish to measure the first N moments under the q -expectation,

$$\langle x^n \rangle_q \equiv \frac{\int dx [\rho(x)]^q x^n}{\int dx [\rho(x)]^q} \quad , \quad (22)$$

in a reliable way (i.e. $n \leq N$). Assume $q > 1$ and consider $\tilde{\mathcal{D}}_{B,\infty}$ as the admissible domain of distributions on $[0, 1]$ (i.e $m = \infty$). It follows that $B > \|g'(x)\rho(x)\|_\infty$. Choose $g(x) = 1 - 1/(1+x)^\phi$ for some $\phi > 0$. Consequently, $g'(x) = \phi(1+x)^{-\phi-1}$. The boundedness condition for the observables immediately requires $\phi > N/(q-1) - 1$ and the decay property for the distributions implies $\bar{q} < 1 + 1/(\phi + 1)$. Inversely, this means that for specific distributions ρ on $[0, \infty]$ it is possible to design domains around these specific ρ where the q -expectation value is uniformly continuous. Again, the discrete case of probabilities p on \mathbb{N}_+ into $[0, \infty]$ can be embedded in the continuous case $[0, \infty]$ using step-functions $\rho_p(x) = p_i$ for $x \in [i-1, i)$. Domains $\tilde{\mathcal{D}}'$ of uniform continuity of the q -expectation value of distributions on $[0, \infty]$ can be pulled back to domains of \mathcal{D}' such that the q -expectation value is uniformly equi-continuous on \mathcal{D}' .

IV. CONCLUSION

To summarize, we have shown that reasonable restrictions on the domain of admissible probability distributions restore uniform continuity for the q -expectation on this domain. Bounds on the size of admissible variations have been given that allow to estimate the overall robustness of the q -expectation under small variations. The practical usefulness of the theorems for estimating the robustness of the q -expectation value with respect to small variations has been discussed.

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Appendix A

This appendix contains the propositions for the proof of theorem (1), the case $0 < q < 1$.

Proposition 1. *Let $D \subset I \equiv [0, 1]$ and $0 < \mathcal{O}$ be a bounded positive function on I . Further, let $\delta\rho$ be a function on I such that $\int_I dx |\delta\rho| = \delta$, then there exists a constant $0 < C_1 < \infty$, such that $\int_D dx |\delta\rho(x)|^q \mathcal{O}(x) \leq C_1 \delta^q$. Furthermore, $C_1 < \|\mathcal{O}\|_\infty |D|^{1-q}$, where $|D| = \int_D dx$.*

Proof. Using the Hölder-inequality find, $\int_D dx |\delta\rho|^q \mathcal{O} \leq (\int_D dx \delta\rho)^q \left(\int_D dx |\mathcal{O}|^{\frac{1}{1-q}} \right)^{1-q}$. Setting $C_1 = \left(\int_D dx |\mathcal{O}|^{\frac{1}{1-q}} \right)^{1-q} < (\|\mathcal{O}\|_\infty^{\frac{1}{1-q}} |D|)^{1-q} = \|\mathcal{O}\|_\infty |D|^{1-q}$, and noting that $(\int_D dx |\delta\rho|)^q \leq (\int_I dx |\delta\rho|)^q = \delta^q$, completes the proof. \square

Proposition 2. Let $D_0 = \{x|\rho(x) = 0\}$ and $D_+ = \{x|\rho(x) > 0\}$ and $r = \int_{D_+} dx \equiv |D_+| = 1 - |D_0|$, then $\int_{D_0} dx |\delta\rho|^q \leq \delta^q(1-r)^{1-q}$, and $\int_{D_+} dx |\delta\rho|^q \leq \delta^q r^{1-q}$.

Proof. Set $\mathcal{O} = 1$ in proposition (1) and use $r = |D_+|$. □

Proposition 3. Let $0 < \rho$ be a non-singular probability distribution on $I = [0, 1]$ and $\delta\rho$ be a variation of the distribution such that $\int_I dx |\delta\rho| = \delta$, and $0 < \rho + \delta\rho$ is positive on I . Further, let $0 < \mathcal{O}$ be a positive bounded observable on I , then there exists a constant $0 < C_2 < \infty$ such that

$$\int_I dx (\rho + \delta\rho)^q \mathcal{O} \leq \left(\int_I dx \rho^q \mathcal{O} \right) + C_2 \delta^q, \quad (23)$$

and $C_2 = \|\mathcal{O}\|_1 + 2q\|\mathcal{O}\|_\infty + C_1 \leq 4\|\mathcal{O}\|_\infty$.

Proof. Let $D_+^- = \{x|0 < \rho(x) \leq \delta\}$ and $D_+^+ = \{x|\delta < \rho(x)\}$, then $\int_I dx (\rho + \delta\rho)^q \mathcal{O} = \int_{D_+^-} dx (\rho + \delta\rho)^q \mathcal{O} + \int_{D_+^+} dx (\rho + \delta\rho)^q \mathcal{O} + \int_{D_0} dx \delta\rho^q \mathcal{O}$. Since a power of $q < 1$ is concave the first term leads to $\int_{D_+^-} dx (\rho + \delta\rho)^q \mathcal{O} \leq \int_{D_+^-} dx (\delta^q + q\delta^{q-1}(\delta\rho + \rho - \delta)) \mathcal{O} \leq \delta^q \left(\int_{D_+^-} dx \mathcal{O} \right) + q\delta^{q-1} \left(\int_{D_+^-} dx |\delta\rho| \mathcal{O} \right) \leq \delta^q (\|\mathcal{O}\|_1 + q\|\mathcal{O}\|_\infty)$. Similarly, the second term leads to $\int_{D_+^+} dx (\rho + \delta\rho)^q \mathcal{O} \leq \int_{D_+^+} dx (\rho^q + q\delta^{q-1}|\delta\rho|) \mathcal{O} \leq \int_I dx \rho^q \mathcal{O} + q\delta^q \|\mathcal{O}\|_\infty$. The third term, that corresponds to the part of the domain where $\rho(x) = 0$, is estimated by proposition (1). Adding all three contributions together leads to the result. □

Proposition 4. Let $0 < \rho$ be a non-singular probability distribution on $I = [0, 1]$ and $\delta\rho$ a variation of the distribution such that, $\int_I dx |\delta\rho| = \delta$, and $0 < \rho + \delta\rho$ is positive on I , then there exists a constant $0 < \tilde{C}_2 < \infty$ such that

$$\int_I dx (\rho + \delta\rho)^q \leq \left(\int_I dx \rho^q \right) + \tilde{C}_2 \delta^q, \quad (24)$$

with $\tilde{C}_2 = 1 + 2q + (1-r)^{1-q} \leq 4$.

Proof. Use proposition (3) and set $\mathcal{O} = 1$ to find $\|\mathcal{O}\|_1 = 1$ and $\|\mathcal{O}\|_\infty = 1$. □

Proposition 5. Under the same conditions as in proposition (3) find that

$$\left(\int_I dx \rho^q \mathcal{O} \right) - C_3 \delta^q \leq \int_I dx (\rho + \delta\rho)^q \mathcal{O}, \quad (25)$$

with $C_3 = \|\mathcal{O}\|_1 + (2q+1)\|\mathcal{O}\|_\infty < 4\|\mathcal{O}\|_\infty$.

Proof. Use proposition (3) with $\rho = (\rho + \delta\rho) - \delta\rho$, i.e. substitute $\rho' = \rho + \delta\rho$ and $\delta\rho' = -\delta\rho$ and adapt D_0 and D_+^\pm to ρ' accordingly. Due to the substitution $\rho \rightarrow \rho'$ the value of $r = |D_+|$ can not be assumed to remain invariant. Choosing the worst possible case, $r = 0$, leads to the result. \square

Proposition 6. *Under the same conditions as in proposition (3) find that*

$$\left(\int_I dx \rho^q \right) - \tilde{C}_3 \delta^q \leq \int_I dx (\rho + \delta\rho)^q \quad , \quad (26)$$

with $\tilde{C}_3 = 2(1 + q) < 4$.

Proof. Use proposition (5) and set $\mathcal{O} = 1$. \square

Proposition 7. *Let $G = \int_I dx \rho(x)^q$ and let $0 < \tilde{\delta}^q = \mu G/4$ for $0 < \mu < 1$. Under the same conditions as in proposition (3), it follows that for all $0 < \delta < \tilde{\delta}$*

$$\frac{\int_I dx \rho^q \mathcal{O}}{\int_I dx (\rho + \delta\rho)^q \mathcal{O}} - 1 \leq \frac{C_3 \delta^q}{\int_I dx \rho^q \mathcal{O} - C_3 \tilde{\delta}^q} \quad . \quad (27)$$

Proof. Use proposition (5) to get $\int_I dx \rho^q \mathcal{O} / \int_I dx (\rho + \delta\rho)^q \mathcal{O} \leq 1 + C_3 \delta^q / \int_I dx (\rho + \delta\rho)^q \mathcal{O}$. Use proposition (5) again on the right hand side to estimate $\int_I dx (\rho + \delta\rho)^q \mathcal{O}$ from below and take the minimal admissible value of this estimate by setting δ^q to $\tilde{\delta}^q$. \square

Proposition 8. *Let $G = \int_I dx \rho(x)^q$ and let $0 < \tilde{\delta}^q = \mu G/4$ for $0 < \mu < 1$. Under the same conditions as in proposition (3) it follows that for all $0 < \delta < \tilde{\delta}$,*

$$\frac{\int_I dx \rho^q}{\int_I dx (\rho + \delta\rho)^q} - 1 \leq \frac{\tilde{C}_3 \delta^q}{G - \tilde{C}_3 \tilde{\delta}^q} \quad . \quad (28)$$

Proof. Repeat the proof of proposition (7) for $\mathcal{O} = 1$, i.e. by using proposition (6) instead of proposition (5). \square

Appendix B

This appendix contains the propositions for the proof of theorem (2), the case $1 < q$. Since $q > 0$, the q -norm $\|f\|_q = (\int_I dx |f(x)|^q)^{1/q}$ is the usual L_q norm.

Proposition 9. Let $B > 0$ be an arbitrary positive constant. Let $\delta\rho$ be functions on $I = [0, 1]$ that are uniformly bounded for some m -norm, i.e. $\|\delta\rho\|_m < B$, where $m > q$. Further let $\|\delta\rho\|_1 = \delta$. Let $0 < \mathcal{O}$ be a positive bounded observable on I , then there exists a constant $0 < R_1 < \infty$, such that

$$\int_I dx |\delta\rho|^q \mathcal{O} \leq R_1 \delta^\gamma, \quad (29)$$

where $\gamma = (m - q)/(m - 1) \leq 1$ and $R_1 = B^{q-\gamma} \|\mathcal{O}\|_\infty$.

Proof. Let γ be a constant $0 < \gamma \leq 1$. $\int_I dx |\delta\rho|^q \mathcal{O} \leq \|\mathcal{O}\|_\infty \|\delta\rho^\gamma |\delta\rho|^{q-\gamma}\|_1 \leq \|\mathcal{O}\|_\infty \|\delta\rho^\gamma\|_{1/\gamma} \|\delta\rho|^{q-\gamma}\|_{1/(1-\gamma)}$ using Hölder's inequality. Now choosing a such that $m = (q - \gamma)/(1 - \gamma)$ and noting that this means $\gamma = (m - q)/(m - 1)$, i.e. $q - \gamma = (q - 1)m/(m - 1)$, we get $\|\delta\rho^\gamma\|_{1/\gamma} \|\delta\rho|^{q-\gamma}\|_{1/(1-\gamma)} = (\|\delta\rho\|_1)^\gamma (\|\delta\rho\|_m)^{q-\gamma} \leq \delta^\gamma B^{q-\gamma}$. \square

Proposition 10. Let $0 < \rho$ be a probability distribution on $I = [0, 1]$, i.e. $\|\rho\|_1 = 1$, with finite q -norm, i.e. $\|\rho\|_q < \infty$. Let $\delta\rho$ be a variation of ρ , i.e. $0 < \rho + \delta\rho$ is positive on I , that has the properties specified in proposition (9). Further let $0 < \tilde{\delta}$ be some positive constant and $\|\delta\rho\|_1 = \delta \leq \tilde{\delta}$ and let $\mathcal{O} > 0$ be a strictly positive bounded observable then there exists a constant $0 < R_2 < \infty$, such that

$$\int_I dx (\rho + \delta\rho)^q \mathcal{O} \leq (1 + R_2 \delta^{\gamma/q}) \int_I dx \rho^q \mathcal{O}. \quad (30)$$

Proof. Since $1 < q$ we first use the Minkowsky inequality and then proposition (9) to get $\int_I dx (\rho + \delta\rho)^q \mathcal{O} \leq ((\int_I dx \rho^q \mathcal{O})^{1/q} + (\int_I dx |\delta\rho|^q \mathcal{O})^{1/q})^q \leq ((\int_I dx \rho^q \mathcal{O})^{1/q} + (R_1 \delta^\gamma)^{1/q})^q \leq \int_I dx \rho^q \mathcal{O} (1 + (R_1 \delta^\gamma / \int_I dx \rho^q \mathcal{O})^{1/q})^q$. Now we note that the minimum for $\int_I dx \rho^q \mathcal{O}$ can be obtained for $\rho^{q-1} \propto \mathcal{O}^{-1}$ and it follows that $\int_I dx \rho^q \mathcal{O} \geq (\|\mathcal{O}^{-1}\|_\infty)^{-1}$. Therefore $\int_I dx (\rho + \delta\rho)^q \mathcal{O} \leq \int_I dx \rho^q \mathcal{O} (1 + z \delta^{\gamma/q})^q$ where $z = (R_1 \|\mathcal{O}^{-1}\|_\infty)^{1/q}$. Now we note that since $q > 1$ for all $\delta < \tilde{\delta}$ it holds that $(1 + z \delta^{\gamma/q})^q \leq 1 + R_2 \delta^{\gamma/q}$ with $R_2 = ((1 + z \tilde{\delta}^{\gamma/q})^q - 1) / \tilde{\delta}^{\gamma/q}$. \square

Proposition 11. Under the same conditions as in proposition (10), there exists a constant $0 < \tilde{R}_2 < \infty$, such that

$$\int_I dx (\rho + \delta\rho)^q \leq (1 + \tilde{R}_2 \delta^{\gamma/q}) \int_I dx \rho^q. \quad (31)$$

Proof. Use proposition (10) and set $\mathcal{O} = 1$. \square

Proposition 12. *Under the same conditions as in proposition (10), and for $0 < \tilde{\delta}$ chosen small enough it holds that $0 < \left(1 - R_2 \tilde{\delta}^{\gamma/q}\right)$ and*

$$(1 - R_2 \delta^{\gamma/q}) \int_I dx \rho^q \mathcal{O} \leq \int_I dx (\rho + \delta \rho)^q \mathcal{O} \quad . \quad (32)$$

Proof. Use proposition (10) with $\rho' = \rho + \delta \rho$ and $\delta \rho' = -\delta \rho$ to get $\int_I dx \rho^q \mathcal{O} \leq (1 + R_2 \delta^{\gamma/q}) \int_I dx (\rho + \delta \rho)^q \mathcal{O}$. Then, divide this result by $(1 + R_2 \delta^{\gamma/q})$ and note that $1/(1+x) > 1-x$ to get the result. In order for $0 \leq \left(1 - R_2 \tilde{\delta}^{\gamma/q}\right)$ to hold simple calculation show that this can be guaranteed by choosing $\tilde{\delta}$ small enough, i.e. $(2^{1/q} - 1)^{q/\gamma} (B^{q-\gamma} \|\mathcal{O}\|_\infty \|\mathcal{O}^{-1}\|_\infty)^{-1/\gamma} > \tilde{\delta}$. \square

Proposition 13. *Under the same conditions as in proposition (10), there exists a constant $0 < \tilde{R}_3 < \infty$, such that*

$$\left(1 - \tilde{R}_2 \delta^{\gamma/q}\right) \int_I dx \rho^q \leq \int_I dx (\rho + \delta \rho)^q \quad . \quad (33)$$

Proof. Use proposition (12) and set $\mathcal{O} = 1$. \square

Proposition 14. *Under the same conditions as in proposition (10) and $0 < \delta \leq \tilde{\delta}$, there exists a constant $0 < R_3 < \infty$, such that*

$$\frac{\int_I dx \rho^q}{\int_I dx (\rho + \delta \rho)^q} - 1 \leq 1 + R_3 \delta^{\gamma/q} \quad . \quad (34)$$

Proof. Use proposition (13) to get $\int_I dx \rho^q / \int_I dx (\rho + \delta \rho)^q \leq 1/(1 - \tilde{R}_2 \delta^{\gamma/q}) \leq 1 + R_3 \delta^{\gamma/q}$ with $R_3 = \left(1/(1 - \tilde{R}_2 \tilde{\delta}^{\gamma/q}) - 1\right) \tilde{\delta}^{-\gamma/q} = \tilde{R}_2/(1 - \tilde{R}_2 \tilde{\delta}^{\gamma/q})$. This completes the proof. \square